1 Problem Session 1 Notes

A very important probability distribution in physics is the Poisson distribution

$$P(n) = \frac{(rt)^n}{n!} e^{-rt} \tag{1.1}$$

The distribution $P(n)$ tells us the probability of observing n discrete events (like molecules colliding with a wall) in time t when the rate of these events is a fixed constant r. You will learn more about this distribution in Problem Set 1 and in our derivation of the ideal gas law from basic principles. Let us first check to see that the distribution is normalized. The random variable n can take on any counting number $n = 0, 1, 2, 3, \ldots, \infty$, so we want to compute

$$\sum_{n=0}^{\infty} P(n) = e^{-rt} \sum_{n=0}^{\infty} \frac{(rt)^n}{n!} \tag{1.2}$$

The summation on the right hand side of this expression is very famous (so much so that you should never forget it). It is the Taylor expansion of the exponential function

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{1.3}$$

If you have not seen this before, I have worked it out at the end of these notes. Therefore setting $x = rt$ shows us that

$$\sum_{n=0}^{\infty} P(n) = e^{-rt} \sum_{n=0}^{\infty} \frac{(rt)^n}{n!} \tag{1.4}$$

$$= e^{-rt} e^{rt} = 1 \tag{1.5}$$

Now let's compute the mean of n

$$\langle n \rangle = \sum_{n=0}^{\infty} nP(n) \tag{1.6}$$

$$= e^{-rt} \sum_{n=0}^{\infty} n \frac{(rt)^n}{n!} \tag{1.7}$$

To solve this problem we need to get a little creative. First we remember that the factorial is given by $n! = n(n-1)(n-2)\cdots 1$ so that

$$\langle n \rangle = e^{-rt} \sum_{n=0}^{\infty} \frac{(rt)^n}{(n-1)!} \tag{1.8}$$

Now, we can make a substitution

$$n' = n - 1 \tag{1.9}$$

to give

$$\begin{aligned}
\langle n \rangle &= e^{-rt} \sum_{n'=-1}^{\infty} \frac{(rt)^{n'+1}}{n'!} && (1.10) \\
&= (rt) e^{-rt} \sum_{n'=-1}^{\infty} \frac{(rt)^{n'}}{n'!} && (1.11) \\
&= (rt) e^{-rt} \left[\frac{(rt)^{-1}}{(-1)!} + \sum_{n'=0}^{\infty} \frac{(rt)^{n'}}{n'!} \right] && (1.12) \\
&= (rt) e^{-rt} \left[\frac{(rt)^{-1}}{(-1)!} + e^{rt} \right] && (1.13)
\end{aligned}$$

Now we're basically done because you may remember from calculus class that the factorial of negative integers is infinite (at least in its magnitude)

$$|(-1)!| = |(-1)(-2)(-3)(-4)\cdots| \to \infty \tag{1.14}$$

so

$$\frac{(rt)^{-1}}{(-1)!} = 0 \tag{1.15}$$

giving us the final answer for the mean

$$\begin{aligned}
\langle n \rangle &= (rt) e^{-rt} \left[0 + e^{rt} \right] && (1.16) \\
&= (rt) && (1.17)
\end{aligned}$$

This makes a lot of sense, because if r is a rate and t is a time then the units work out

$$\left[\frac{\#\ \text{events}}{\text{time}} \right] \times \left[\frac{\text{time}}{1} \right] = \#\ \text{events} \tag{1.18}$$

2 Taylor Series of the Exponential Function

The Taylor series of a function $f(x)$ about the point x_0 is given by

$$\begin{aligned}
f(x) &= f(x_0) + \frac{df(x_0)}{dx}(x - x_0) + \frac{1}{2!} \frac{d^2 f(x_0)}{dx^2}(x - x_0)^2 + \cdots && (2.19) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{d^n f(x)}{dx^n} \right)_{x=x_0} (x - x_0)^n && (2.20)
\end{aligned}$$

For the exponential function, we have for $x_0 = 0$

$$\begin{aligned}
\left. \frac{d}{dx} e^x \right|_{x=0} &= e^0 = 1 \\
\left. \frac{d^2}{dx^2} e^x \right|_{x=0} &= e^0 = 1 && (2.21) \\
&\vdots \\
\left. \frac{d^n}{dx^n} e^x \right|_{x=0} &= e^0 = 1
\end{aligned}$$

$$(2.22)$$

so the Taylor series is given by

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots \tag{2.23}$$

we can rewrite this as

$$e^x = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \tag{2.24}$$

since $0! = 1$, and finally using summation notation

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{2.25}$$